



# TECHNICAL NOTE

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ANALYSIS OF A LINEAR SYSTEM FOR VARIABLE-THRUST CONTROL  
IN THE TERMINAL PHASE OF RENDEZVOUS

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## SUMMARY

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A linear system for applying thrust to a ferry vehicle in the terminal phase of rendezvous with a satellite is analyzed. This system requires that the ferry thrust vector per unit mass be variable and equal to a suitable linear combination of the measured position and velocity vectors of the ferry relative to the satellite. The variations of the ferry position, speed, acceleration, and mass ratio are examined for several combinations of the initial conditions and two basic control parameters analogous to the undamped natural frequency and the fraction of critical damping. Upon making a desirable selection of one control parameter and requiring minimum fuel expenditure for given terminal-phase initial conditions, a simplified analysis in one dimension practically fixes the choice of the remaining control parameter. The system can be implemented by an automatic controller or by a pilot.

## INTRODUCTION

The problem of the rendezvous of space vehicles is one of increasing significance in space research and engineering. References 1 to 14, which are typical of the available literature in this field, contain analyses of various aspects of the rendezvous problem. In these papers attention has been given mainly to the problem of most immediate interest, namely, that of the rendezvous of an earth-launched ferry vehicle with a satellite or space station in orbit about the earth.

From the standpoint of thrust application, terminal-phase rendezvous systems include (a) those which utilize one or more constant-thrust rocket motors, and (b) systems with continuously variable thrust. Terminal-phase rendezvous systems employing continuously variable thrust have been analyzed. (For example, see refs. 8, 13, and 14.) Systems of type (b), another of which is considered in this paper, can be approached in practice by systems of type (a); thus, the throttability limitations of existing rocket motors may not be a serious obstacle to the design of a system which, in its simplest form, would utilize one or more throttleable

motors. Methods for approaching a system of type (b) by systems of type (a) are illustrated schematically in figure 1.

In this paper, thrust which is continuously variable in magnitude and direction will be assumed to be available. The basic principle of the thrust control system to be employed requires that the instantaneous vector acceleration of the ferry be made equal to the sum of two vectors, which are suitable multiples of the position and velocity vectors, respectively, of the ferry relative to the satellite. Thus, the acceleration vector of the ferry is made to be a suitable linear combination of its relative position and velocity vectors. The three vectors are then coplanar. For the analysis of this system, a nonrotating satellite-centered coordinate system will be used. (See fig. 2.) The choice of the control parameters, that is, determining the linear combination of the position and velocity vectors that should be used, is examined herein. Moreover, the effects of the initial conditions and of gravitation on the terminal-phase motion and the rocket fuel expenditure are discussed.

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#### SYMBOLS

$a$	distance between satellite and ferry at start of terminal phase of rendezvous
$\vec{F}$	ferry thrust vector
$G$	Newton's universal gravitational constant, $6.670 \times 10^{-11}$ newton-meter <sup>2</sup> /kg <sup>2</sup> or $3.438 \times 10^{-8}$ lb-ft <sup>2</sup> /slug <sup>2</sup>
$\vec{i}, \vec{j}, \vec{k}$	orthogonal unit vectors
$m$	ferry mass
$m_0$	ferry mass at start of terminal phase of rendezvous
$m_\infty$	limit of $m$ as $t$ approaches $\infty$
$M$	mass of earth, $5.975 \times 10^{24}$ kg or $4.094 \times 10^{23}$ slugs
$R$	distance between satellite and ferry
$t$	time
$T$	satellite orbital period
$V_1, V_2$	ferry relative velocity components at start of terminal phase of rendezvous

$v_1, v_2$	nondimensional values of $V_1, V_2$ (referred to $\omega_0 a$ )
$v_{ex}$	effective exit speed of propulsive exhaust gases
$\vec{x}$	ferry position vector in a coordinate system with origin at satellite and axes always parallel to lines fixed in an inertial frame
$x_1, x_2, x_3$	components of vector $\vec{x}$
$\alpha_1$	value of $\ddot{x}_1$ at $t = 0$
$\xi$	control parameter analogous to fraction of critical damping
$\vec{\eta}$	ferry position vector in a coordinate system with origin at center of earth and axes oriented as for $\vec{x}$ above
$\theta$	angle, $\tan^{-1} \frac{x_2}{x_1}$
$\mu$	mass variable, $\left(\frac{m}{m_0}\right)^{v_{ex}/\omega_0 a}$
$\vec{\xi}$	satellite position vector in a coordinate system with origin at center of earth and axes oriented as for $\vec{x}$
$\xi$	distance between satellite and center of earth
$\xi_1, \xi_2, \xi_3$	components of vector $\vec{\xi}$
$\omega_0$	control parameter analogous to undamped natural frequency
$\omega = \omega_0 \sqrt{1 - \xi^2}$	when $\xi < 1$
$\Omega = \omega_0 \sqrt{\xi^2 - 1}$	when $\xi > 1$

Dots over symbols indicate differentiation with respect to time.

## SOME GENERAL CONSIDERATIONS ON THE TERMINAL PHASE OF RENDEZVOUS

In order to have a basis for ideas, it may be useful to define the start and end of the terminal phase of rendezvous as the times at which, respectively:

(a) Radar or other contact has been established between ferry and satellite to permit measurements of relative position and velocity and based on these measurements, final-approach thrusting is initiated to reduce the relative distance and velocity to low values (for example, 100 feet and 5 feet per second).

(b) Relative position and velocity have been reduced to sufficiently low values to permit the initiation of docking.

Before proceeding to the discussion of the thrust-application system which is the subject of this paper, brief consideration will be given to some simple concepts which are useful in rendezvous studies when the effect of gravitation on the motion of the ferry relative to the satellite is a minor one.

If the velocity of one vehicle is measured with respect to a stable platform in the other vehicle, the analysis presented in reference 15 (pages 5 to 10) can be applied directly to the motion of the ferry relative to the satellite provided the gravitational field is approximately central. In particular, the analysis indicates that gravitation may be of secondary importance (depending upon the thrust level, the apparent gravitational acceleration, and so forth) in its effect on the relative motion in the terminal phase of a given rendezvous situation. If this is the case, rendezvous systems can be subjected to approximate analyses which neglect the gravitational effect. The limitations of simplified studies of this type can subsequently be appraised in a number of ways, for example, by computing the time integral of the apparent gravitational acceleration and comparing this integral with a speed characteristic of the ferry's approach. In this way, more detailed and accurate studies of trajectories, fuel consumption, and so forth may be deferred until the final stage of a design program.

Consider a nonrotating coordinate system (see fig. 2) with origin at the satellite (target) and assume that gravitational effects on the relative motion are negligible. In this reference frame, the primary forces acting on the ferry are the thrust and orientation-control forces. The ferry speed (relative to this reference frame) at the start of the terminal phase puts a lower bound on the required fuel expenditure. Rendezvous with this minimum expenditure of fuel can clearly be accomplished in a finite time only if the ferry relative velocity vector points directly toward the origin (the position of the satellite in the

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reference frame). In general, then, it is advantageous to start the terminal phase of rendezvous with the ferry velocity vector (1) directed as nearly as possible toward the origin, and (2) of just sufficient magnitude (speed) to permit rendezvous to be accomplished in a specified time without the necessity of extra expenditure of fuel to hasten the closure.

For example, in a nonrotating, earth-centered reference frame, a ferry may be launched into an elliptical orbit which osculates the satellite orbit from inside the latter. (See fig. 3(a).) If the flight is planned so that the ferry arrives at the oscule (osculation point) somewhat ahead of the satellite and so that the terminal-phase thrusting begins at this time, the expenditure of fuel in the terminal phase principally completes the process of bringing the speed of the ferry up to the speed of the satellite. On the other hand, in the case of a ferry orbit which osculates the satellite orbit from outside the latter (fig. 3(b)), it would be preferable to have the satellite reach the oscule ahead of the ferry, at which time the terminal-phase thrusting is initiated and is utilized mainly to bring the speed of the ferry down to the speed of the satellite. In both examples, small variations of the direction of the thrust vector which might be required for rendezvous maneuvering would add little to the fuel expenditure. However, the system to be analyzed is not restricted to these situations.

#### A LINEAR SYSTEM FOR THRUST APPLICATION

The basic requirement for the terminal approach, the near nullification of the relative position and velocity vectors, calls for measurements of either these vectors or an equivalent set of variables.

One of the simplest types of terminal-phase rendezvous systems utilizing continuously variable thrust is one in which the thrust vector per unit mass is made a linear combination of the relative position and velocity vectors of the ferry. (See fig. 2.) The following analysis of this type of system at first neglects the effect of gravitation on the relative motion. Subsequently, the central-field gravitational effect is examined in some cases of current interest. Trajectory perturbations due to drag, electric or magnetic fields, solar radiation pressure, and so forth are neglected throughout the paper. Similarly, problems concerned with the orientation of the ferry as a rigid body are not considered.

In a rectangular Cartesian coordinate system with origin at the satellite and with axes which maintain constant directions with respect to an inertial frame, that is, with respect to the fixed stars, let the position vector of the ferry be denoted by

$$\vec{x} = x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k} \quad (1)$$

where  $\vec{i}$ ,  $\vec{j}$ , and  $\vec{k}$  are unit vectors along the coordinate axes. It should be noted that the negative of  $\vec{x}$  is the position vector of the satellite with respect to a stable platform or a star-oriented system of axes in the ferry. Under the simplifying assumptions of the preceding paragraph, the vector equation of relative motion is

$$\frac{d^2 \vec{x}}{dt^2} = \frac{\vec{F}}{m} \quad (2)$$

where  $m$  is the mass of the ferry and the vector  $\vec{F}$  is the thrust acting on it.

In the terminal-phase rendezvous system considered here, thrust is applied in accordance with the equation

$$\frac{\vec{F}}{m} = -2\zeta\omega_0 \frac{d\vec{x}}{dt} - \omega_0^2 \vec{x} \quad (3)$$

where  $\omega_0$  and  $\zeta$  are suitably chosen constants. If the right-hand side of equation (3) is considered as the desired thrust per unit mass and the left-hand side as that which is actually applied, it is clear that the equation can be only approximately satisfied in practice. The action which equation (3) calls for can be effected in a number of ways by an automatic controller or by a pilot. The limiting case of a perfect controller is considered in the analysis that follows, that is, equation (3) is assumed to be exact.

Equations (2) and (3) yield

$$\frac{d^2 \vec{x}}{dt^2} + 2\zeta\omega_0 \frac{d\vec{x}}{dt} + \omega_0^2 \vec{x} = \vec{0} \quad (4)$$

which is the equation of motion of the ferry relative to the satellite for the system under consideration. Equation (4) is equivalent to the equation of motion of a harmonic oscillator in three dimensions with damping proportional to the velocity. The second term, or damping term, in equation (4) may be thought of as the "slowing" part, whereas the third term, or restoring term, could be called the "zeroing" part. The latter, however, is only a tendency. Except in the trivial case with the initial conditions (at  $t = 0$ )

$$\vec{x} = \vec{0}$$



$$\frac{d\vec{x}}{dt} = \vec{0}$$

equation (4) yields no solution corresponding to rendezvous in the precise sense

$$\vec{x} = \vec{0}$$

$$\frac{d\vec{x}}{dt} = \vec{0}$$

(at  $t = \text{Some finite time}$ ) for these final conditions imply  $\vec{x} = \vec{0}$  for all  $t$  (including  $t = 0$ ), which is the trivial case mentioned. The proof, which is a simple consequence of the theory of linear differential equations with constant coefficients, is omitted.

Rendezvous in the practical sense, that is, the near nullification of  $\vec{x}$  and  $d\vec{x}/dt$  within a reasonable time, can be accomplished with the system described by equation (4). Moreover, the system possesses certain desirable features besides the simplicity of its analytical expression. These, together with the most questionable feature, the assumption of continuously variable thrust, will be discussed subsequently.

It follows from its vector nature that equation (4) is invariant under rotations of the coordinate system. For convenience, then, let the unit vector  $\vec{i}$  have the direction and sense which the position vector  $\vec{x}$  has at  $t = 0$ , the start of the terminal phase. Thus, at  $t = 0$

$$\vec{x} = a\vec{i} \quad (a > 0) \quad (5)$$

It is also convenient to choose the unit vector  $\vec{j}$  so that (see fig. 4), at  $t = 0$ ,

$$\frac{d\vec{x}}{dt} = V_1\vec{i} + V_2\vec{j} \quad (V_2 \geq 0) \quad (6)$$

This choice of  $\vec{j}$  is evidently unique unless  $V_2 = 0$ . In this case the motion is confined to the line of  $\vec{i}$ . Otherwise, the trajectory determined by equations (4), (5), and (6) is confined to the plane of  $\vec{i}$  and  $\vec{j}$ , that is,  $x_3 = 0$  for all  $t$ .

The parameters  $\omega_0$  and  $\xi$  in equation (4) are easily recognized as the undamped natural frequency and the fraction of critical damping, respectively, of the problem of the damped harmonic oscillator. Thus, multiplication by  $\omega_0$  renders  $t$  dimensionless.

Solution for  $\zeta < 1$ 

The solution of equation (4) subject to the initial conditions (5) and (6) can, for  $\zeta < 1$ , be expressed in the dimensionless form

$$\left. \begin{aligned} \frac{x_1}{a} &= \left[ \cos \omega t + (\zeta + v_1) \frac{\sin \omega t}{\sqrt{1 - \zeta^2}} \right] e^{-\zeta \omega_0 t} \\ \frac{x_2}{a} &= v_2 \frac{\sin \omega t}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_0 t} \end{aligned} \right\} \quad (7)$$

where

$$\left. \begin{aligned} \omega &= \omega_0 \sqrt{1 - \zeta^2} \\ v_1 &= \frac{v_1}{\omega_0 a} \\ v_2 &= \frac{v_2}{\omega_0 a} \end{aligned} \right\} \quad (8)$$

The corresponding velocity components are given by

$$\left. \begin{aligned} \frac{\dot{x}_1}{\omega_0 a} &= \left[ v_1 \cos \omega t - (1 + \zeta v_1) \frac{\sin \omega t}{\sqrt{1 - \zeta^2}} \right] e^{-\zeta \omega_0 t} \\ \frac{\dot{x}_2}{\omega_0 a} &= \left( v_2 \cos \omega t - \zeta v_2 \frac{\sin \omega t}{\sqrt{1 - \zeta^2}} \right) e^{-\zeta \omega_0 t} \end{aligned} \right\} \quad (9)$$

where the dots refer to differentiation with respect to time. The acceleration components are, then,

$$\left. \begin{aligned} \frac{\ddot{x}_1}{\omega_0^2 a} &= \left\{ -(1 + 2\zeta v_1) \cos \omega t + \left[ \zeta + (2\zeta^2 - 1)v_1 \right] \frac{\sin \omega t}{\sqrt{1 - \zeta^2}} \right\} e^{-\zeta \omega_0 t} \\ \frac{\ddot{x}_2}{\omega_0^2 a} &= \left[ -2\zeta v_2 \cos \omega t + (2\zeta^2 - 1)v_2 \frac{\sin \omega t}{\sqrt{1 - \zeta^2}} \right] e^{-\zeta \omega_0 t} \end{aligned} \right\} \quad (10)$$

Equations (7), (9), and (10) correspond to the underdamped ( $\zeta < 1$ ) harmonic oscillator.

Solution for  $\zeta = 1$

For critical damping ( $\zeta = 1$ ), the characteristic equation of equation (4) has equal roots. In this case, the solution of equation (4), again subject to equations (5) and (6), takes the form

$$\left. \begin{aligned} \frac{x_1}{a} &= [1 + (1 + v_1)\omega_0 t] e^{-\omega_0 t} \\ \frac{x_2}{a} &= v_2 \omega_0 t e^{-\omega_0 t} \end{aligned} \right\} \quad (11)$$

with velocity components

$$\left. \begin{aligned} \frac{\dot{x}_1}{\omega_0 a} &= [v_1 - (1 + v_1)\omega_0 t] e^{-\omega_0 t} \\ \frac{\dot{x}_2}{\omega_0 a} &= v_2 (1 - \omega_0 t) e^{-\omega_0 t} \end{aligned} \right\} \quad (12)$$

and acceleration components

$$\left. \begin{aligned} \frac{\ddot{x}_1}{\omega_0^2 a} &= [-(1 + 2v_1) + (1 + v_1)\omega_0 t] e^{-\omega_0 t} \\ \frac{\ddot{x}_2}{\omega_0^2 a} &= v_2 (-2 + \omega_0 t) e^{-\omega_0 t} \end{aligned} \right\} \quad (13)$$

Solution for  $\zeta > 1$

Finally, for the overdamped case ( $\zeta > 1$ ), the solution has the form

$$\left. \begin{aligned} \frac{x_1}{a} &= \left[ \cosh \Omega t + (\zeta + v_1) \frac{\sinh \Omega t}{\sqrt{\zeta^2 - 1}} \right] e^{-\zeta \omega_0 t} \\ \frac{x_2}{a} &= v_2 \frac{\sinh \Omega t}{\sqrt{\zeta^2 - 1}} e^{-\zeta \omega_0 t} \end{aligned} \right\} \quad (14)$$

where  $\Omega = \omega_0 \sqrt{\zeta^2 - 1}$ . The corresponding velocity components are

$$\left. \begin{aligned} \frac{\dot{x}_1}{\omega_0 a} &= \left[ v_1 \cosh \Omega t - (1 + \zeta v_1) \frac{\sinh \Omega t}{\sqrt{\zeta^2 - 1}} \right] e^{-\zeta \omega_0 t} \\ \frac{\dot{x}_2}{\omega_0 a} &= v_2 \left( \cosh \Omega t - \zeta \frac{\sinh \Omega t}{\sqrt{\zeta^2 - 1}} \right) e^{-\zeta \omega_0 t} \end{aligned} \right\} \quad (15)$$

and acceleration components are

$$\left. \begin{aligned} \frac{\ddot{x}_1}{\omega_0^2 a} &= \left\{ -(1 + 2\zeta v_1) \cosh \Omega t + \left[ \zeta + (2\zeta^2 - 1)v_1 \right] \frac{\sinh \Omega t}{\sqrt{\zeta^2 - 1}} \right\} e^{-\zeta \omega_0 t} \\ \frac{\ddot{x}_2}{\omega_0^2 a} &= v_2 \left[ -2\zeta \cosh \Omega t + (2\zeta^2 - 1) \frac{\sinh \Omega t}{\sqrt{\zeta^2 - 1}} \right] e^{-\zeta \omega_0 t} \end{aligned} \right\} \quad (16)$$

### Mass Variation

For any value of  $\zeta$ , momentum considerations yield the well-known rocket equation

$$m \left| \frac{\ddot{x}}{x} \right| dt = -v_{ex} dm \quad (17)$$

where  $v_{ex}$  is the effective exit speed of the propulsive exhaust gases. With the separation of variables and integration, equation (17) yields the dimensionless equation for the mass  $m$  of the vehicle at time  $t$ ,

$$\left( \frac{m}{m_0} \right)^{v_{ex}/\omega_0 a} = \exp \left[ - \int_0^{\omega_0 t} \left( \frac{\left| \frac{\ddot{x}}{x} \right|}{\omega_0^2 a} \right) d(\omega_0 t) \right] \quad (18)$$

where  $m_0$  denotes the vehicle mass at  $t = 0$ . This equation will be used in the subsequent study of fuel consumption. The quantity on the left-hand side of equation (18) is called the mass variable and it is independent of  $v_{ex}$ .

## CRITICALLY DAMPED CASE IN ONE DIMENSION

Before a general examination of the problem is made, a restricted class of cases will be considered. The earlier discussion concerned with figure 3 indicated the fuel-consumption advantage to be gained from rendezvous arranged so that the relative velocity of the ferry is directed as nearly as possible toward the satellite, that is,  $v_1 < 0$  and  $v_2 = 0$ . Accordingly, this one-dimensional motion (in the absence of gravitation) will now be examined for the intermediate case where  $\zeta = 1$ . Fuel consumption is minimized in these cases by requiring that  $\ddot{x}_1$  not change sign during the terminal phase. Hence, it is of interest to focus attention on those cases for which  $\ddot{x}_1 \geq 0$  for all  $t \geq 0$ . This will be true if (and only if)

$$-1 \leq v_1 \leq -\frac{1}{2} \quad (19)$$

which follows readily from the first of equations (13). Thus, for the purpose of design, a reasonable choice is

$$v_1 = -\frac{3}{4} \quad (20)$$

which is equivalent to the frequency choice

$$\omega_0 = -\frac{4}{3} \frac{V_1}{a} \quad (21)$$

For this choice, equations (11), (12), and (13) reduce to

$$\left. \begin{aligned} \frac{x_1}{a} &= \left(1 + \frac{1}{4} \omega_0 t\right) e^{-\omega_0 t} \\ \frac{\dot{x}_1}{V_1} &= \left(1 + \frac{1}{3} \omega_0 t\right) e^{-\omega_0 t} \\ \frac{\ddot{x}_1}{\alpha_1} &= \left(1 + \frac{1}{2} \omega_0 t\right) e^{-\omega_0 t} \end{aligned} \right\} \quad (22)$$

where  $\alpha_1$  is the value of  $\ddot{x}_1$  at  $t = 0$ , and equation (18) reduces to

$$\frac{m}{m_0} = \exp \left\{ \frac{V_1}{v_{ex}} \left[ 1 - \left(1 + \frac{1}{3} \omega_0 t\right) e^{-\omega_0 t} \right] \right\} \quad (23)$$

Since  $\omega_0 > 0$  (when  $V_1 < 0$ ), equation (23) becomes, in the limit as  $t \rightarrow \infty$ ,

$$\frac{m_{\infty}}{m_0} = e^{V_1/v_{ex}} \quad (24)$$

or, for  $-V_1 \ll v_{ex}$ ,

$$\frac{m_{\infty}}{m_0} \approx 1 - \frac{-V_1}{v_{ex}} \quad (25)$$

where

$$m_{\infty} = \lim_{t \rightarrow \infty} m \quad (26)$$

Equations (22) are plotted in dimensionless form in figure 5. It is evident from the equations themselves that the proportion

$$\frac{x_1}{a} : \frac{\dot{x}_1}{V_1} : \frac{\ddot{x}_1}{\alpha_1}$$

which initially has the value 1:1:1, approaches the value (referred to norm  $x_1/a$ )

$$1 : \frac{4}{3} : 2$$

with increasing time; however, this ratio is only very roughly obtained within reasonable times.

As a specific example, with  $\xi = 1$ ,  $v_1 = -3/4$ ,  $v_2 = 0$ , let  $a = 100,000$  feet and  $-V_1 = 500$  feet per second. Then,  $\omega_0 = 1/150$  per second and  $\alpha_1 = 2.2$  feet per second per second. Finally, at time  $t = 15$  minutes, the distance has been reduced to  $x_1 = 620$  feet, the ferry is approaching at speed  $-\dot{x}_1 = 3.7$  feet per second, and the acceleration due to thrust has dropped to  $\ddot{x}_1 = 0.022$  foot per second per second. If the rocket motor were shut down at this point, the ferry would coast<sup>1</sup> to the satellite in an additional  $\frac{620}{3.7}$  seconds or 2.8 minutes. Finally, if  $v_{ex} = 10,000$  feet per second, the mass loss due to fuel consumption is only about 5 percent of the initial mass. The magnitudes in this example, with the possible exception of the 100:1 reduction in

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<sup>1</sup>The first two of equations (22) can be used to show that the coasting time  $\frac{x_1}{-\dot{x}_1}$  is nearly independent of the thrust cut-off time  $t_c$ .

As  $t_c$  increases from 0 to  $\infty$ , the coasting time decreases from  $\frac{4}{3\omega_0}$  to  $\frac{1}{\omega_0}$ .

thrust acceleration, seem to be reasonable for the terminal phase of rendezvous with a near earth satellite. Moreover, the effect of the apparent gravitational acceleration can be shown to be of secondary importance in this case in both the thrusting and coasting phases.

Since a rocket motor which can be throttled to 0.01 of maximum thrust is not likely to be developed within the foreseeable future, the implementation of the present rendezvous system for the specific case just discussed would require either a suitable combination of perhaps two to three variable-thrust motors or a scheme approaching variable-thrust performance. (See fig. 1.) A less extreme throttling ratio would result in the case considered, of course, if a higher coasting speed than 3.7 feet per second were tolerated. The discussion in the next section will clarify this statement.

In this specific example, the system affords adequate time to prepare for docking or, in the event of some malfunction, to take emergency actions that may be required. Figure 5 shows that most of the relative-speed reduction occurs early in the thrusting phase. Therefore, since the relative velocity  $\dot{\vec{x}} = \dot{x}_1 \vec{i} + \dot{x}_2 \vec{j}$  will ordinarily not be in line with the origin in the more general case for which  $v_2 \neq 0$ , the probability of collision at high relative speed due to rocket failure must be less for this system than for systems utilizing higher thrust levels late in the terminal phase. This property, together with some other matters of practical importance, will be made clearer by the more detailed study that follows.

#### DETAILED STUDY WITH GRAVITATION NEGLECTED

Equations (7), (9) to (16), and (18) have been used to compute the quantities

$$\frac{|\vec{x}|}{a} \equiv \sqrt{\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{a}\right)^2}$$

$$\theta \equiv \tan^{-1} \frac{x_2}{x_1}$$

$$\frac{|\dot{\vec{x}}|}{\omega_0 a} \equiv \sqrt{\left(\frac{\dot{x}_1}{\omega_0 a}\right)^2 + \left(\frac{\dot{x}_2}{\omega_0 a}\right)^2}$$

$$\frac{|\ddot{\vec{x}}|}{\omega_0^2 a} \equiv \sqrt{\left(\frac{\ddot{x}_1}{\omega_0^2 a}\right)^2 + \left(\frac{\ddot{x}_2}{\omega_0^2 a}\right)^2}$$

$$\mu \equiv \left(\frac{m}{m_0}\right)^{v_{ex}/\omega_0 a}$$

for dimensionless times in the interval 0,14. All combinations of the parameters

$$v_1 = -1.0, -0.5$$

$$v_2 = 0, 0.25, 0.5$$

$$\zeta = 0.7, 1.0, 1.3$$

were used. The values of  $v_1$  which were chosen are the extremes found in the one-dimensional analysis above. (See expression (19).) The results are shown in figure 6. The mass ratio  $m/m_0$  can be found from the mass variable  $\mu$  by making use of the identity

$$\frac{m}{m_0} = \exp\left(\frac{\omega_0 a}{v_{ex}} \log_e \mu\right)$$

Since  $\omega_0 a \ll v_{ex}$  for cases of practical interest, the mass ratio  $m/m_0$  is much nearer unity than is the corresponding value of  $\mu$ . For example, in the numerical example of the previous section,  $m/m_0 \approx 0.95$ , whereas  $\mu \approx \exp(-3/4) \approx 0.47$ . In the absence of gravitation, the optimal final ( $t = \infty$ ) value of  $\mu$  is, by equation (18),  $\exp(-\sqrt{v_1^2 + v_2^2})$ . This ideal value can be achieved, in principle, by applying thrust to reduce the initial velocity to zero; infinitesimal impulses are then used to complete the rendezvous (infinite time being required unless  $v_2 = 0$ ).

Consider first those cases in figure 6 for which  $\zeta = 1.0$  and  $\zeta = 1.3$ . It is noted that the relative distance, speed, and acceleration decrease approximately exponentially with time. Moreover, the total change in azimuth angle is less than  $180^\circ$  in each case. Finally, the ferry mass loss is considerably closer to the ideal value in each of these cases than in the corresponding case for  $\zeta = 0.7$ .

For those cases in figure 6 for which  $\zeta = 0.7$ , the distance, azimuth angle, speed, and acceleration all change much less regularly. Furthermore, the total change in azimuth angle is in each case much

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larger than for the corresponding ones for which  $\xi = 1.0$  and  $\xi = 1.3$ . In particular, for  $\xi = 0.7$  and  $v_2 = 0$ , collision must occur unless it is averted by applying more thrust than the control system requires. (See fig. 6, parts (a) and (b), where collision corresponds to an abrupt change of azimuth angle.)

In the interest of brevity, the variations of the direction angles of the vectors  $\dot{\vec{x}}$  and  $\vec{x}$  have not been included in figure 6. They are readily found, if needed, from the appropriate equations in a manner similar to that used to compute the azimuth angle  $\theta$  as a function of time.

Although figure 6 displays certain disadvantages of choosing  $\xi = 0.7$  it fails to give a clear comparison of the remaining cases,  $\xi = 1.0$  and  $\xi = 1.3$ , with one another. For this reason, the ratio of the ferry acceleration to its initial value is plotted in figure 7 both as a function of the ratio of distance to initial distance and as a function of the ratio of speed to initial speed. Figure 7 shows that, in most cases of practical interest, the range of thrust acceleration required to reduce either distance or speed to a specified fraction of the initial value is substantially less for  $\xi = 1.0$  than for  $\xi = 1.3$ .

The application of the results of figures 6 and 7 to a specific rendezvous problem necessitates more detailed considerations than the broad features which have been mentioned. When the basic rendezvous requirement is taken into account, together with the rate at which distance and speed are reduced, the smoothness of change of the variables concerned, and the mass expended in propulsion, the choice of  $\xi = 1.0$  seems, in the absence of extreme design requirements, to represent a reasonable starting point for examining the applicability of the present system. Thus, the one-dimensional analysis presented in the preceding section is particularly significant.

#### EFFECT OF GRAVITATION

The effect of gravitation will be examined for cases in which the primary attraction is that of the earth. Let  $\vec{\xi}$  and  $\vec{\eta}$  be the position vectors of satellite and ferry, respectively, in a rectangular Cartesian coordinate system with origin at the center of the earth and axes which correspond in direction and sense to the unit vectors of equation (1). The equation of motion of the satellite is, neglecting oblateness, and so forth,

$$\frac{d^2\vec{\xi}}{dt^2} = - \frac{GM}{\xi^2} \frac{\vec{\xi}}{\xi} \quad (23)$$

where  $M$  denotes the mass of the earth and  $G$  denotes Newton's universal gravitational constant. Similarly, the ferry's equation of motion is

$$\frac{d^2 \vec{\eta}}{dt^2} = - \frac{GM}{\eta^2} \frac{\vec{\eta}}{\eta} + \frac{\vec{F}}{m} \quad (24)$$

Since

$$\vec{\eta} = \vec{\xi} + \vec{x} \quad (25)$$

it follows that

$$\frac{d^2 \vec{x}}{dt^2} = \frac{d^2 \vec{\eta}}{dt^2} - \frac{d^2 \vec{\xi}}{dt^2} \quad (26)$$

Equations (23) to (26) yield the equation of relative motion

$$\frac{d^2 \vec{x}}{dt^2} = \frac{\vec{F}}{m} + GM \left( \frac{\vec{\xi}}{\xi^3} - \frac{\vec{\xi} + \vec{x}}{|\vec{\xi} + \vec{x}|^3} \right) \quad (27)$$

For the system of thrust application expressed in equation (3), equation (27) becomes

$$\frac{d^2 \vec{x}}{dt^2} = -2\zeta\omega_0 \frac{d\vec{x}}{dt} - \omega_0^2 \vec{x} + GM \left( \frac{\vec{\xi}}{\xi^3} - \frac{\vec{\xi} + \vec{x}}{|\vec{\xi} + \vec{x}|^3} \right) \quad (28)$$

In terms of its components,

$$\vec{\xi} = \xi_1 \vec{i} + \xi_2 \vec{j} + \xi_3 \vec{k} \quad (29)$$

where the unit vectors  $\vec{i}$ ,  $\vec{j}$ ,  $\vec{k}$  are the same as those in equation (1).

Numerical integrations of equation (28) have been carried out for a circular satellite orbit described by the parametric equations

$$\left. \begin{aligned} \xi_1 &= \xi \sin \frac{2\pi t}{T} \\ \xi_2 &= -\xi \cos \frac{2\pi t}{T} \\ \xi_3 &= 0 \end{aligned} \right\} \quad (30)$$

where the orbital period  $T$  is given by

$$\frac{T}{2\pi} = \left( \frac{\xi^3}{GM} \right)^{1/2} \quad (31)$$

The radius selected was  $\xi = 2.354 \times 10^7$  feet, which corresponds to an altitude of approximately  $2.64 \times 10^6$  feet or 500 statute miles. Consideration was restricted to in-plane cases, that is,  $x_3 \equiv 0$ . Consequently, the ferry's position is given by the range (distance)

$$R = \left( x_1^2 + x_2^2 \right)^{1/2} \quad (32)$$

and the angle

$$\theta = \tan^{-1} \frac{x_2}{x_1} \quad (33)$$

The values 0.7 and 1.0 were used for the parameter  $\zeta$  and values 0.005, 0.01, and 0.04 per second for the parameter  $\omega_0$ .

The results of the numerical computations are presented in figures 8 to 12. The dashed curves, for which gravitation was neglected, are presented for comparison. The agreement is good for these sets of initial conditions. Thus, the simplified approach (gravitational effects being neglected) which was adopted earlier in this paper is justified for the approximate analysis of comparable cases.

#### CONCLUDING REMARKS

The linear system which has been considered for variable ferry-vehicle thrust control provides most of the relative-speed reduction early in the terminal phase. Position and velocity measurements relative to a stable platform or its equivalent are required.

The system's fuel economy was found to be good for values of the control parameter  $\zeta$  of 1.0 and 1.3. In particular, it approaches the optimum as the normal component  $V_2$  of the initial relative velocity approaches zero.

Thrust which is variable in magnitude as well as in direction is required in this system. Ways of meeting this requirement have been

indicated, but the degree of success in a given situation may depend upon the early development of rocket motors which can be used in this application.

The required range of variability of thrust acceleration is substantially less in most cases for  $\zeta = 1.0$  than for  $\zeta = 1.3$ . Since a value of 0.7 for  $\zeta$  was found to be undesirable in other respects, the choice of a value of  $\zeta$  in the neighborhood of 1.0 is favorably indicated by the results which have been presented. The selection of the remaining control parameter  $\omega_0$  should be consistent with the design initial conditions.

Langley Research Center,  
National Aeronautics and Space Administration,  
Langley Field, Va., June 29, 1961.

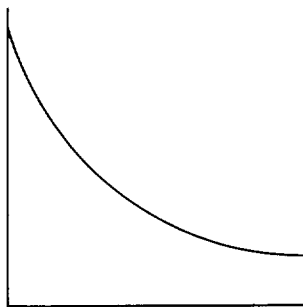
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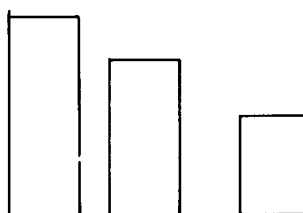
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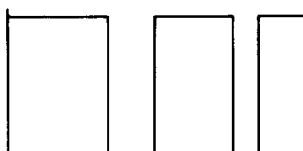
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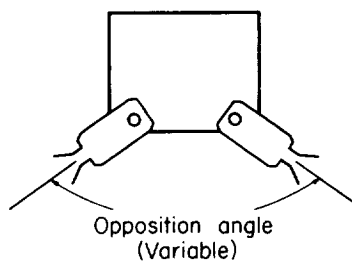
(a) Thrust program for a system of type (b).



(b) Approximation by system of type (a) for which thrust interval is fixed, but number of rockets fired is variable.



(c) Approximation by system of type (b) for which thrust magnitude is fixed, but duration is variable.



(d) Variable resultant thrust achieved by combination of constant-thrust motors with variable opposition angle. (Generally poor fuel economy.)

Figure 1.- Schematic illustration of methods for approaching a system of type (b) by systems of type (a).

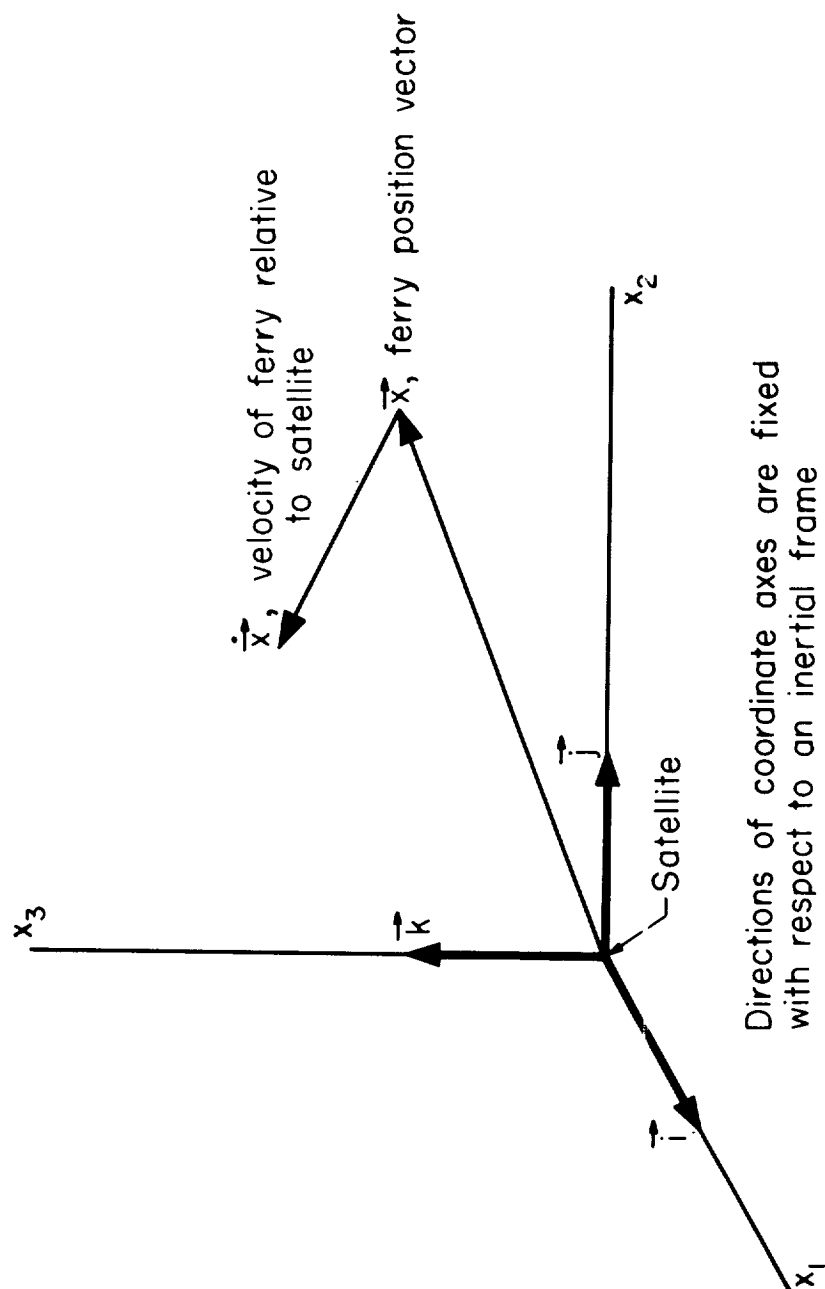
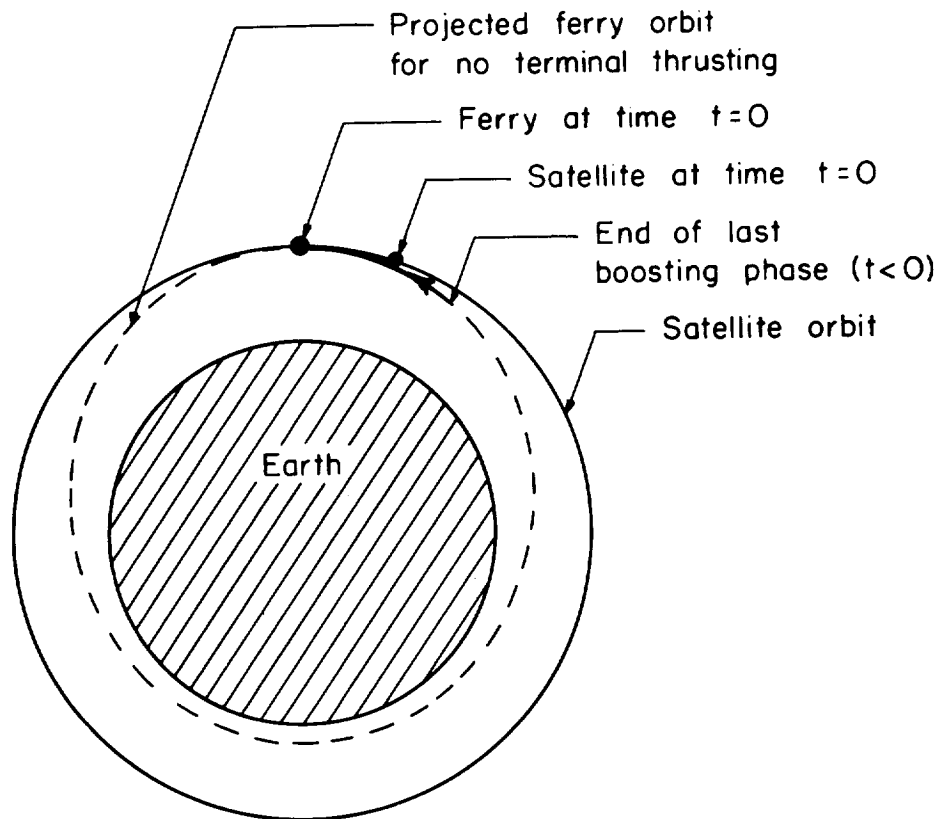


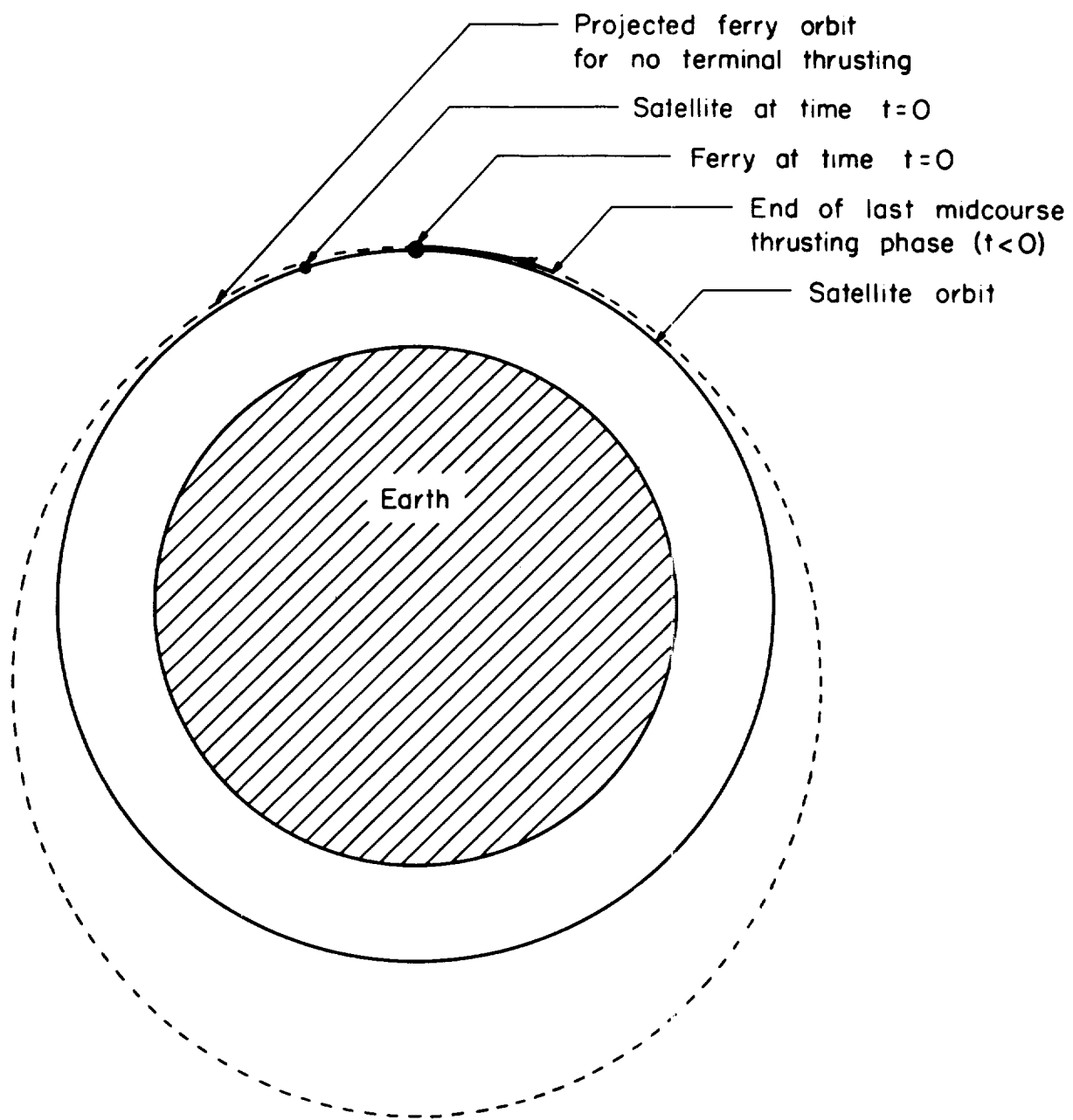
Figure 2.- Position and velocity of ferry in a nonrotating (star-oriented) coordinate system with origin at satellite.





(a) Ferry launched from earth.

Figure 3.- Schematic illustration of desired situation at start of terminal-phase thrusting (for time  $t = 0$ ).



(b) Ferry approaching from space.

Figure 3.- Concluded.

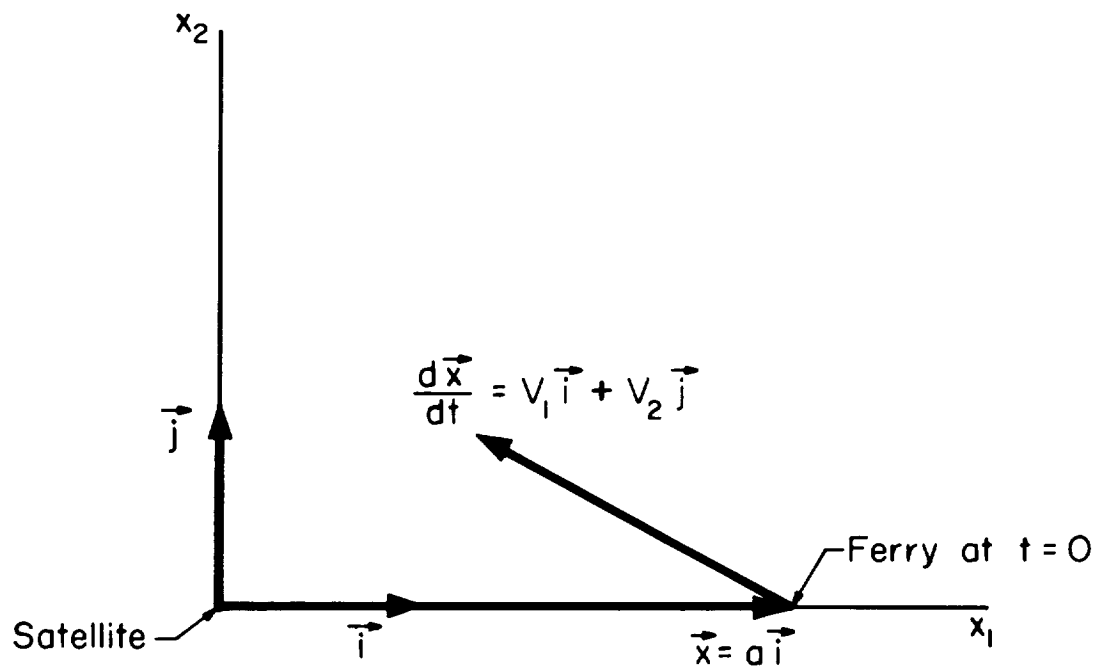


Figure 4.- Position and velocity vectors of ferry relative to satellite at start of terminal phase of rendezvous.

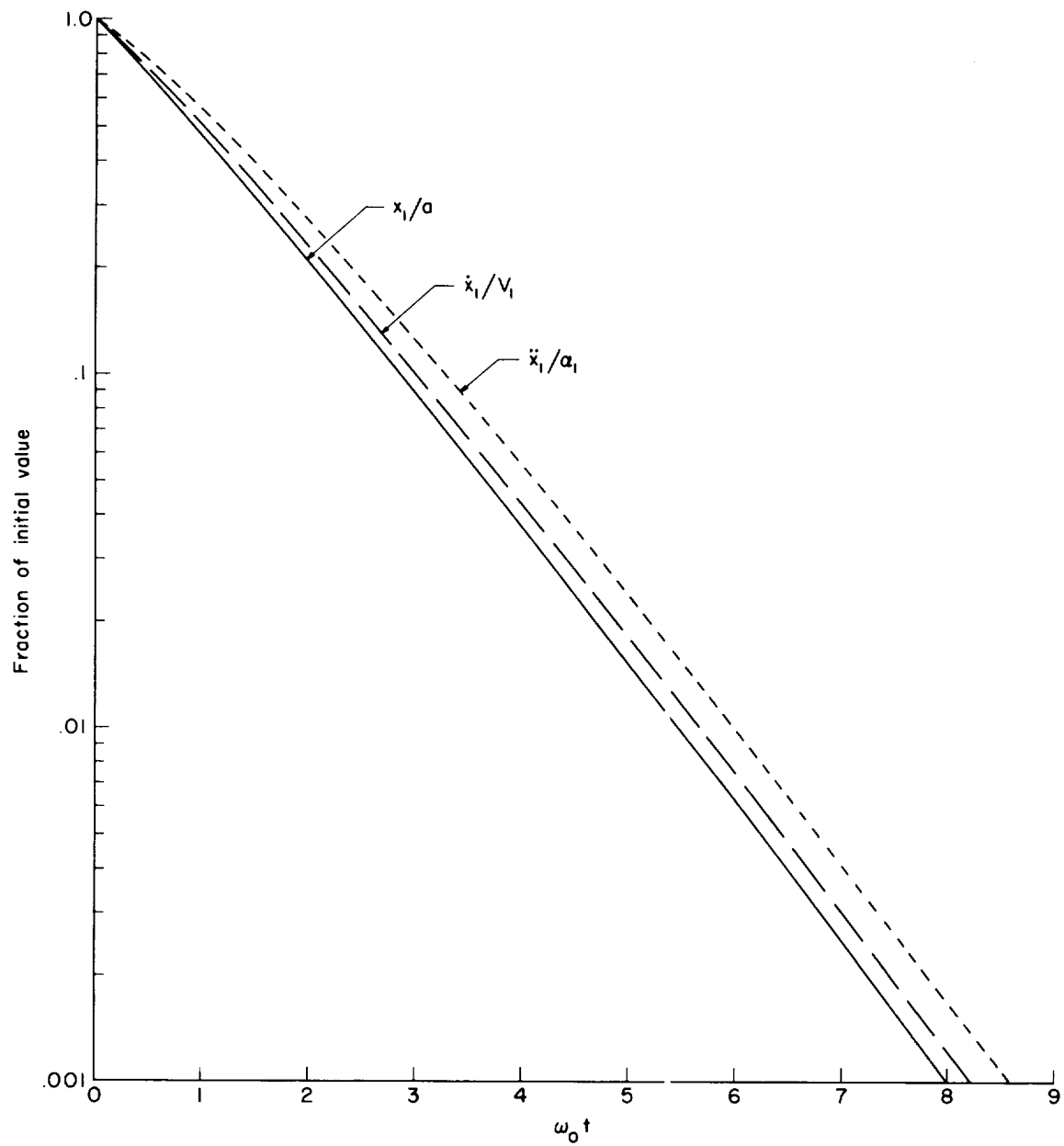


Figure 5.- Variations of distance, speed, and acceleration with time for  $\zeta = 1$ ,  $v_1 = -3/4$ ,  $v_2 = 0$ . See equations (22).

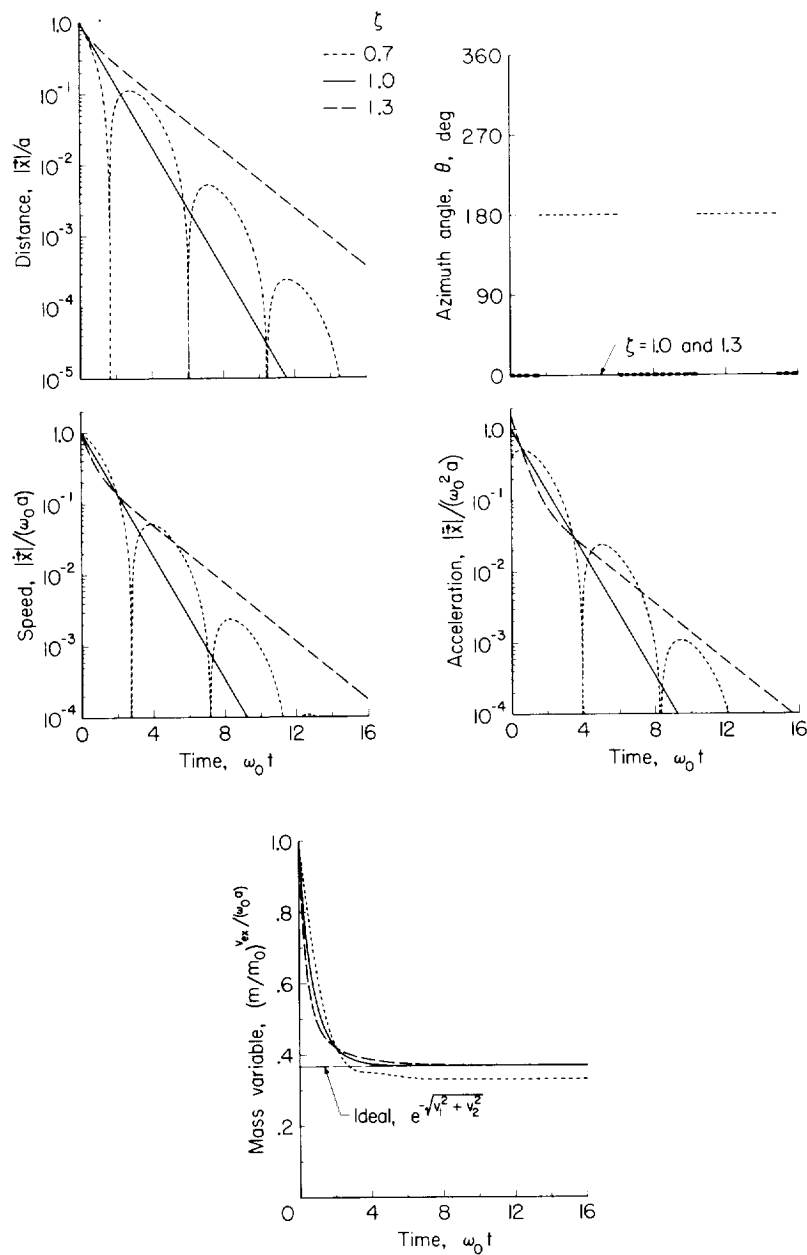
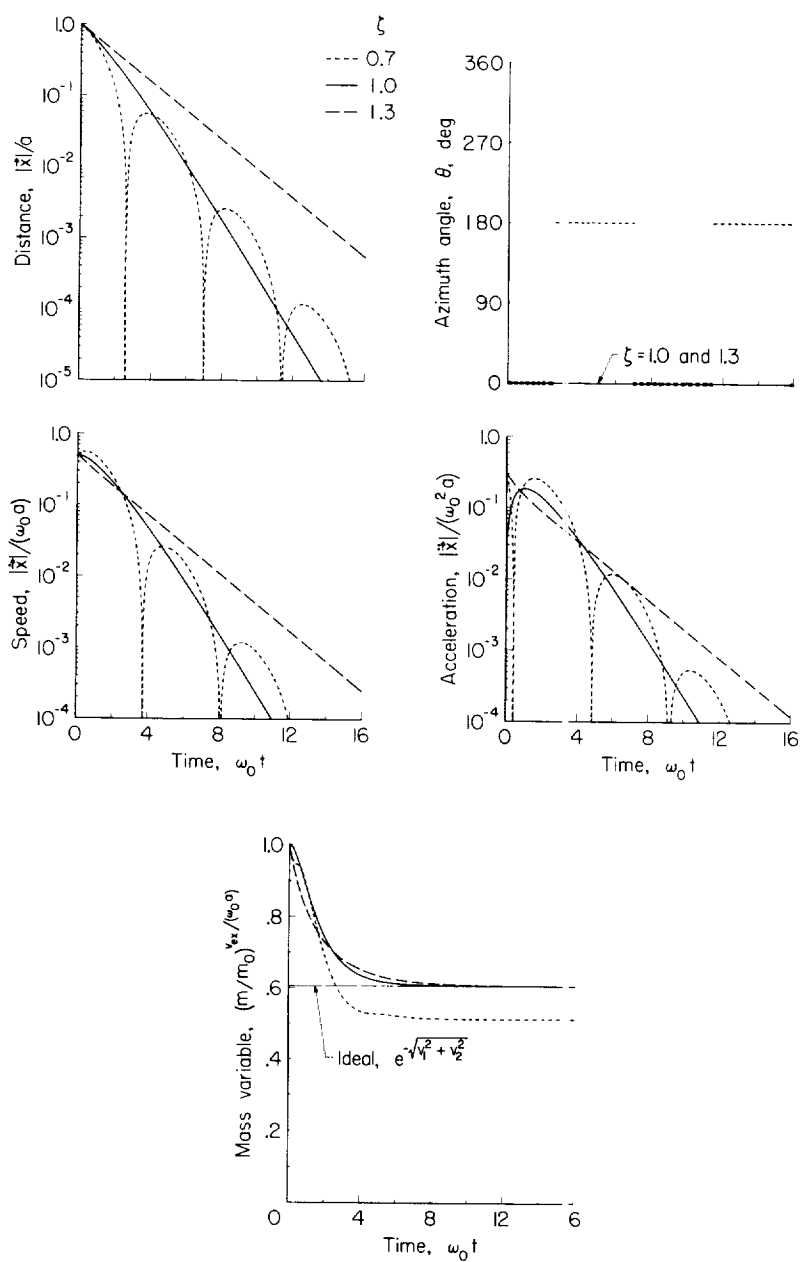
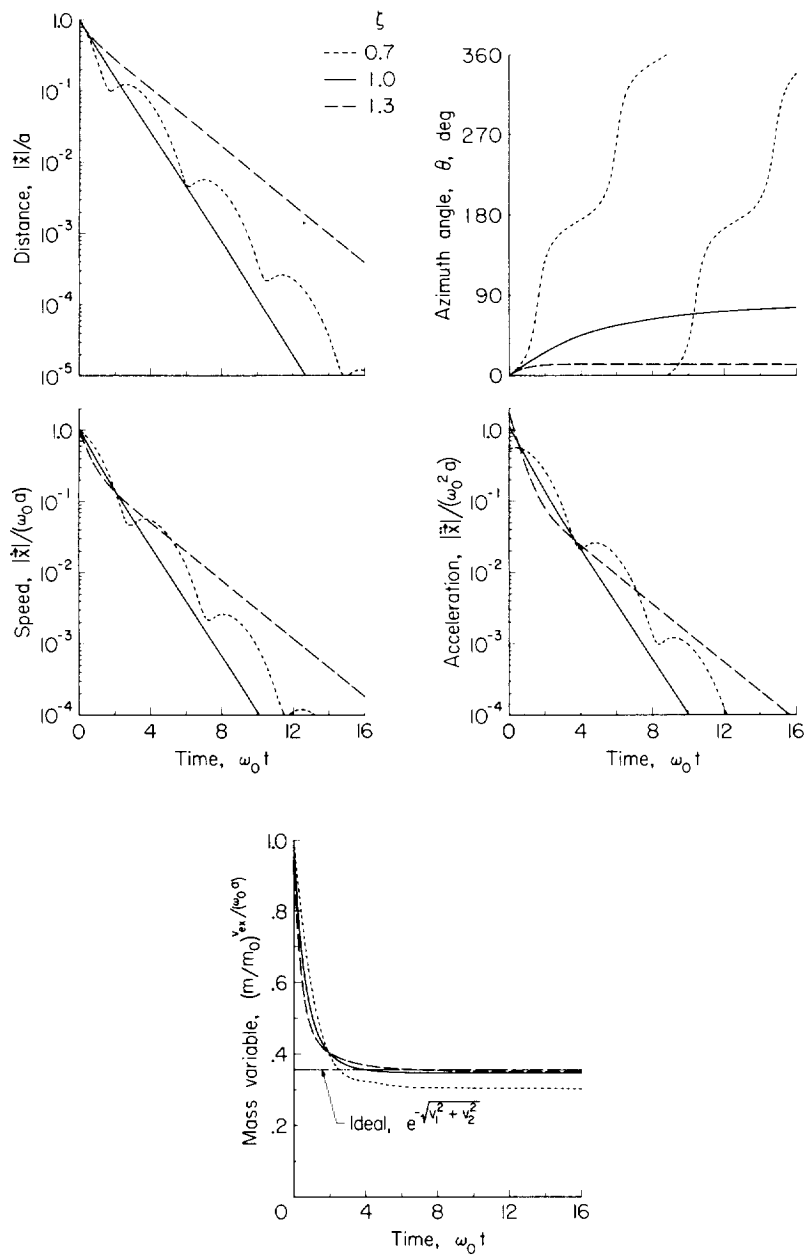
(a)  $V_1 = -1.0$ ;  $V_2 = 0$ .

Figure 6.- Graphs of the dimensionless variables relative position (distance and angle), speed, acceleration, and mass variable against time for various values of  $\zeta$  and initial relative velocity  $V_1, V_2$ .



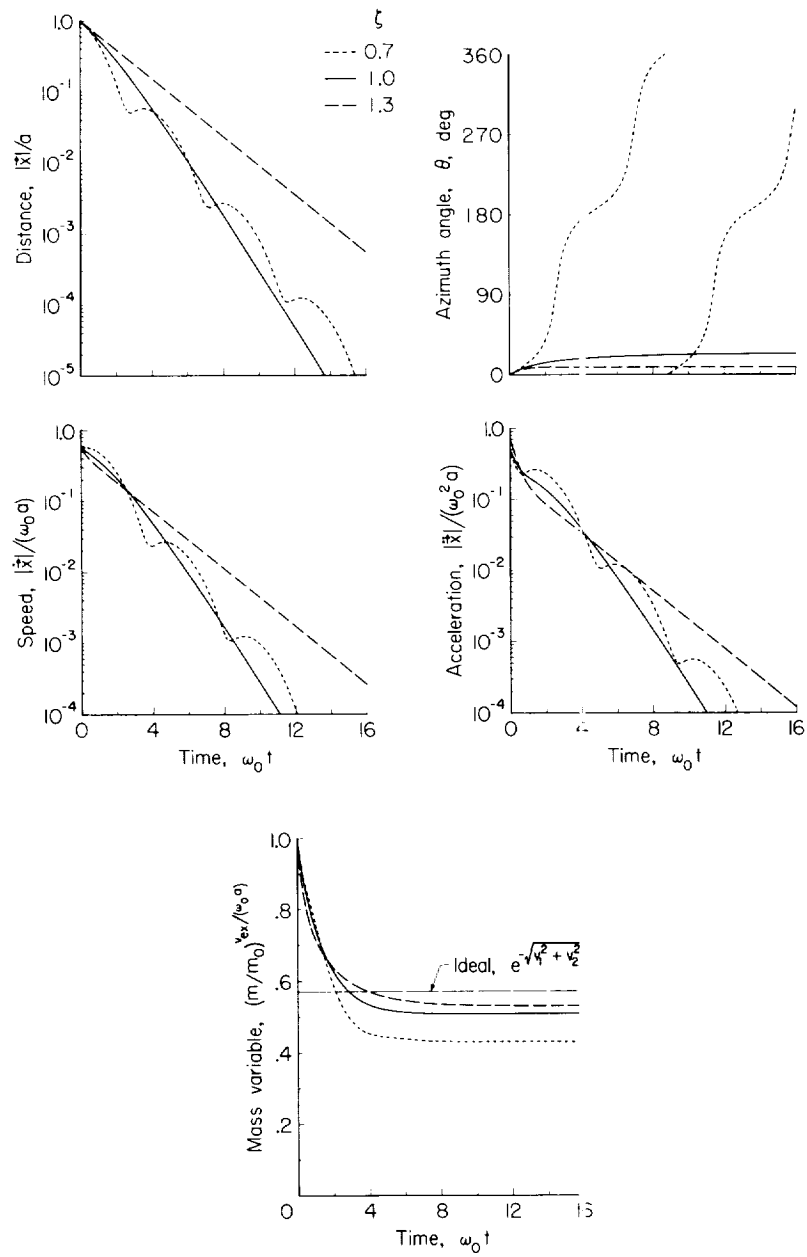
(b)  $V_1 = -0.5$ ;  $V_2 = 0$ .

Figure 6.- Continued.



(c)  $v_1 = -1.0$ ;  $v_2 = 0.25$ .

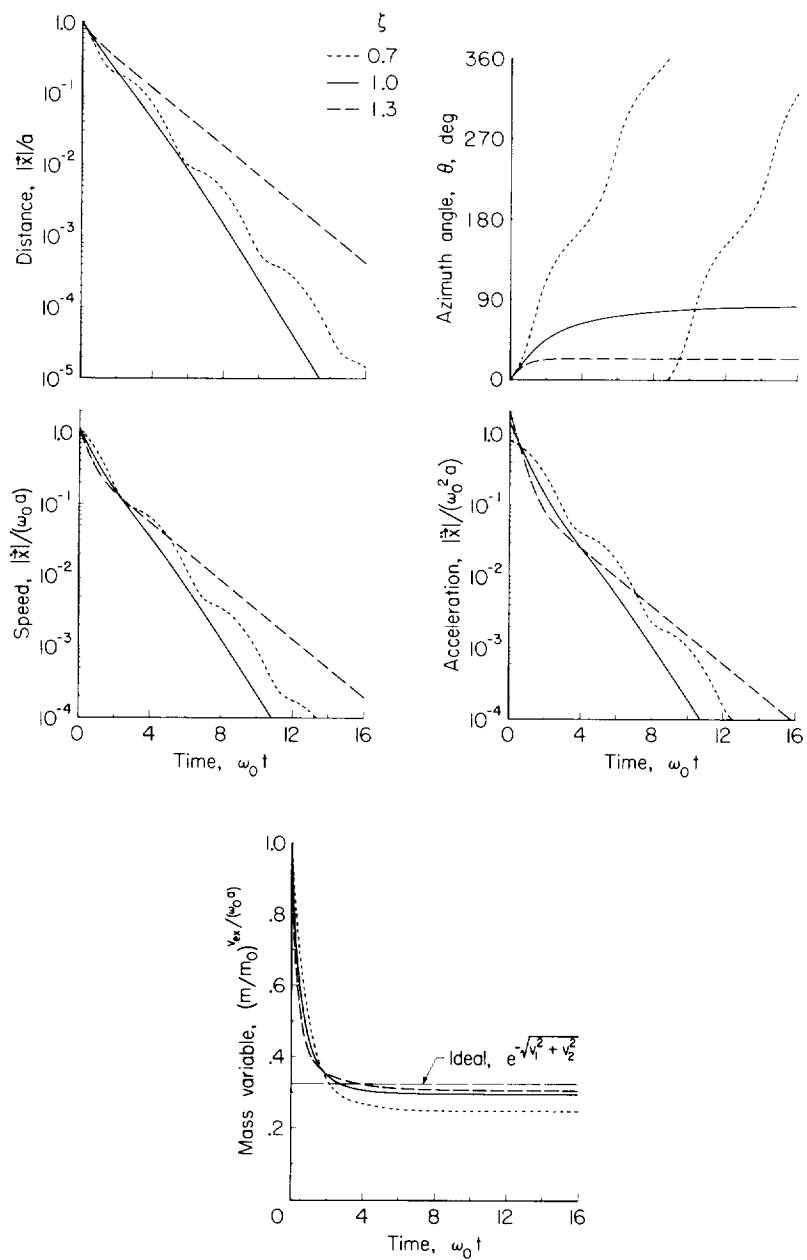
Figure 6.- Continued.



(d)  $v_1 = -0.5$ ;  $v_2 = 0.25$ .

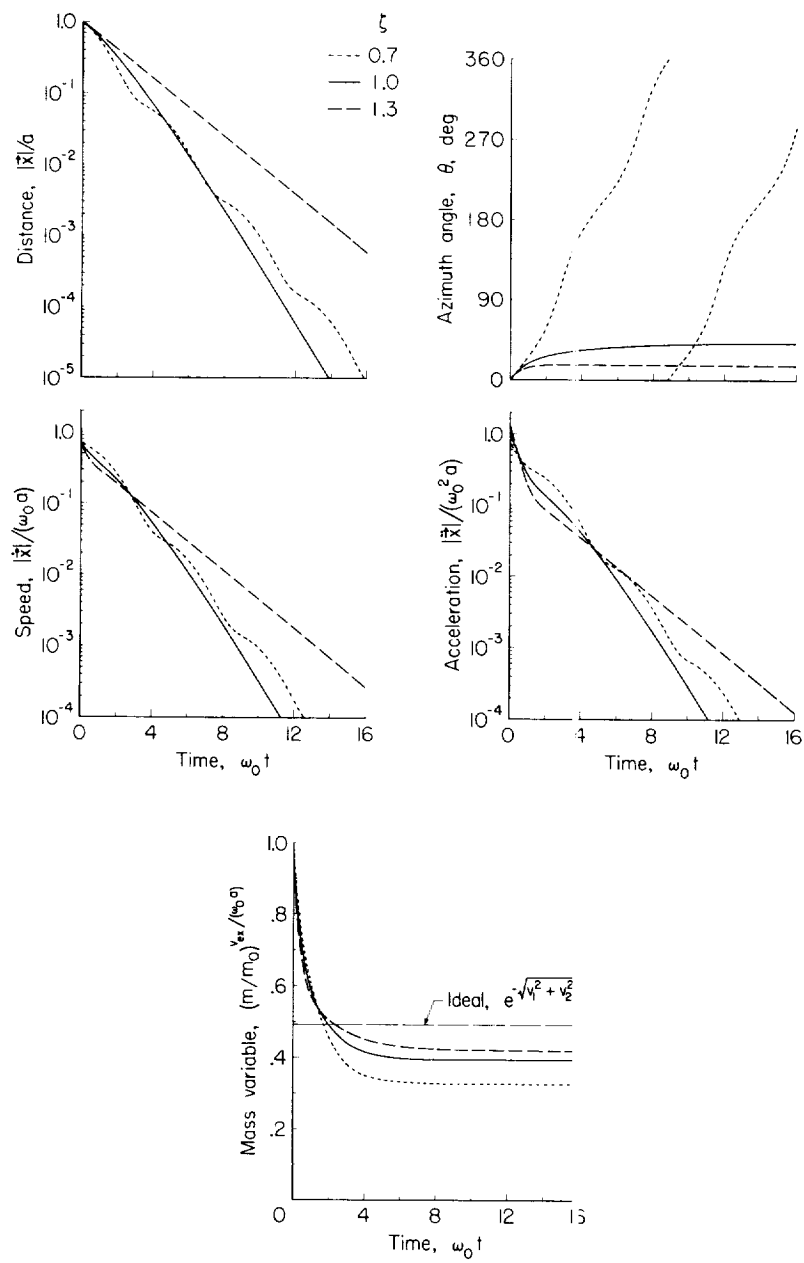
Figure 6.- Continued.





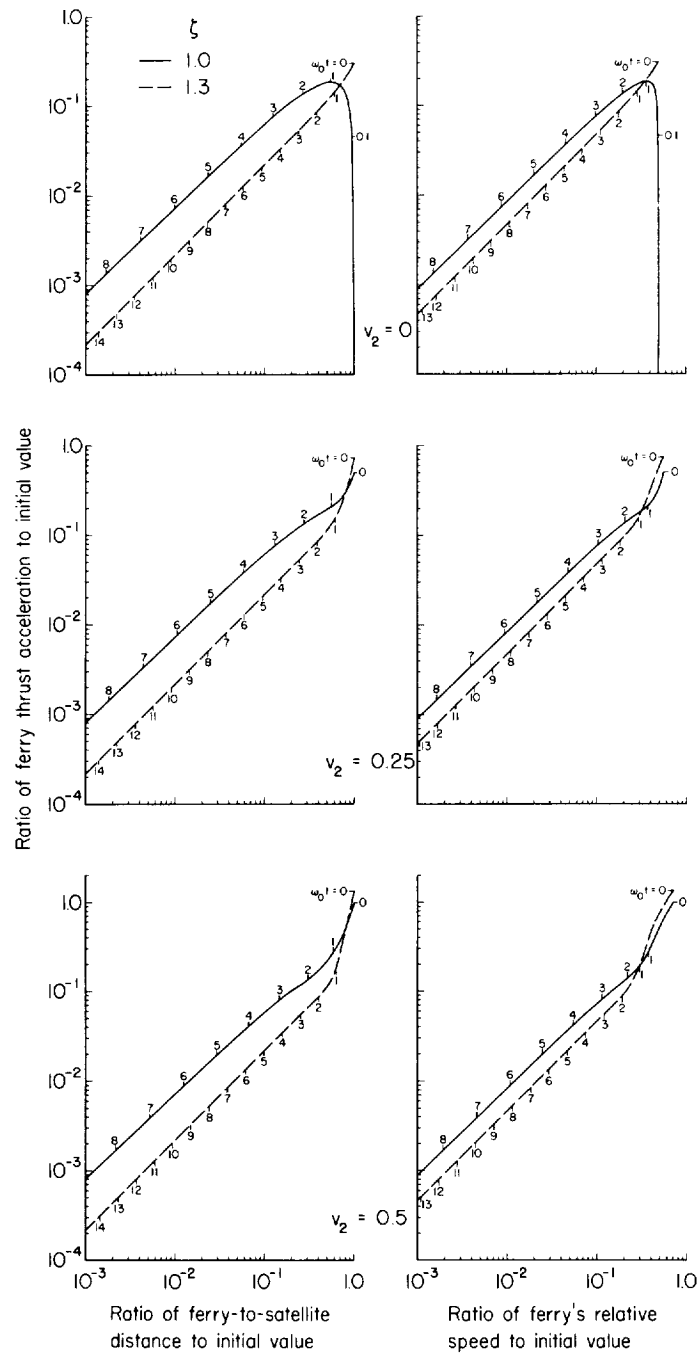
(e)  $v_1 = -1.0$ ;  $v_2 = 0.5$ .

Figure 6.- Continued.



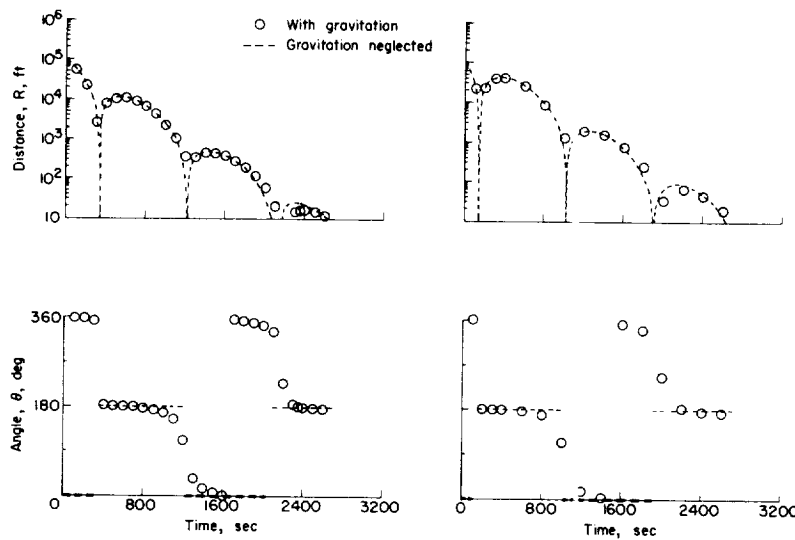
(f)  $V_1 = -0.5$ ;  $V_2 = 0.5$ .

Figure 6.- Concluded.



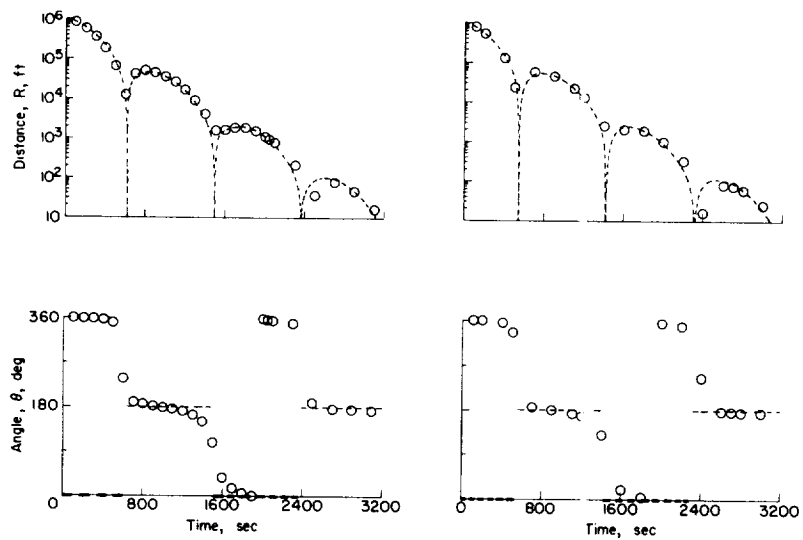
(a)  $v_1 = -0.5$ .

Figure 7.- Graphs of thrust acceleration against distance and speed, all referred to their initial ( $t = 0$ ) values, for  $\zeta = 1.0$  and  $\zeta = 1.3$ .



(e)  $x_1 = 100,000$  feet;  
 $\dot{x}_1 = -500$  feet per second.

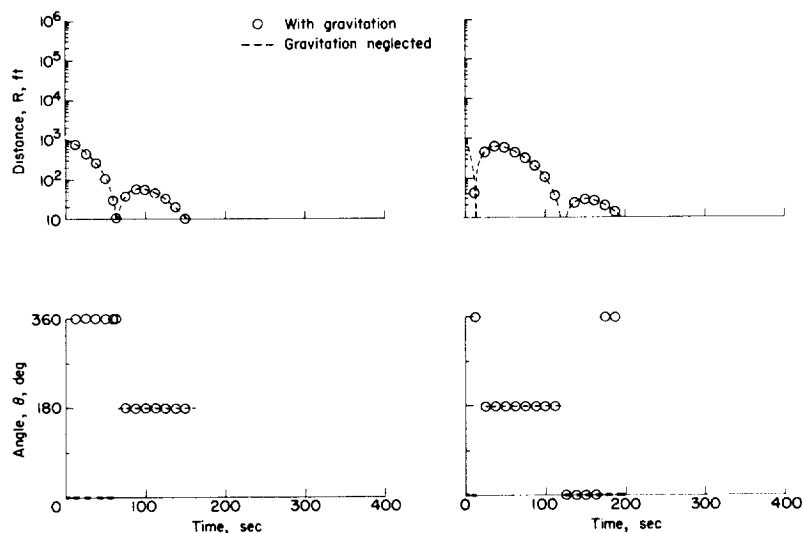
(f)  $x_1 = 100,000$  feet;  
 $\dot{x}_1 = -1,000$  feet per second.



(g)  $x_1 = 1,000,000$  feet;  
 $\dot{x}_1 = -1,000$  feet per second.

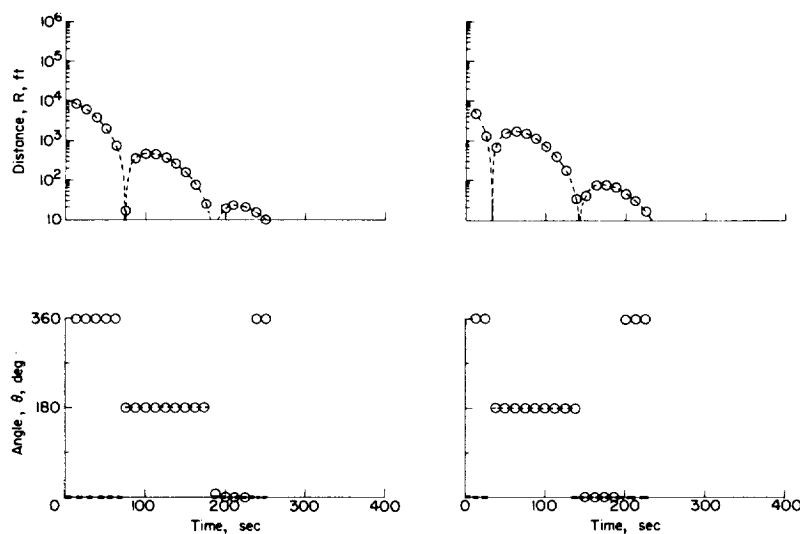
(h)  $x_1 = 1,000,000$  feet;  
 $\dot{x}_1 = -2,000$  feet per second.

Figure 8.- Concluded.



(a)  $x_1 = 1,000$  feet;  
 $\dot{x}_1 = -20$  feet per second.

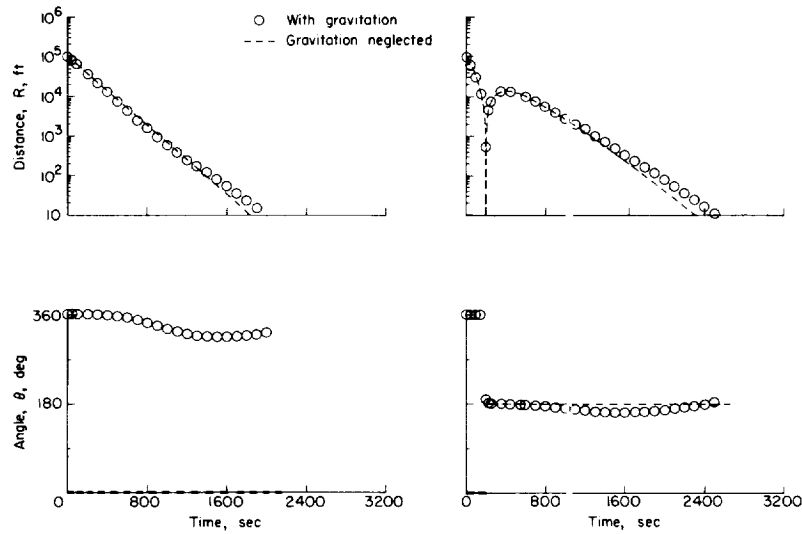
(b)  $x_1 = 1,000$  feet;  
 $\dot{x}_1 = -100$  feet per second.



(c)  $x_1 = 10,000$  feet;  
 $\dot{x}_1 = -100$  feet per second.

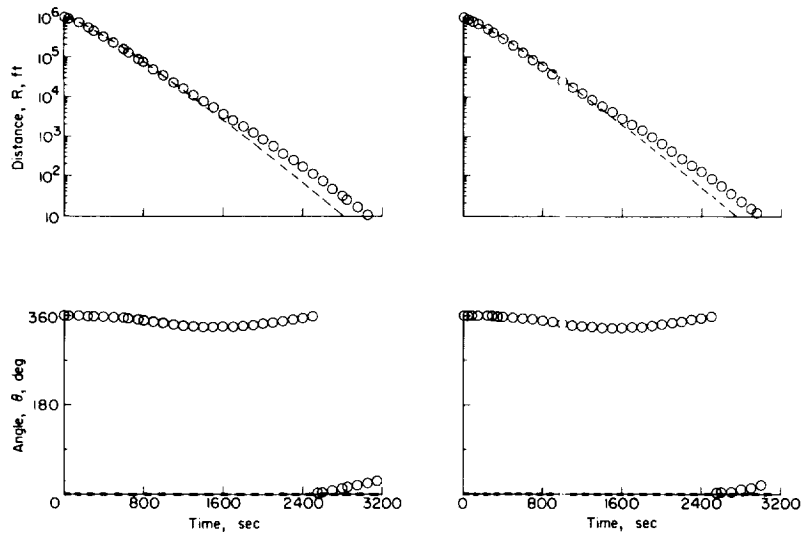
(d)  $x_1 = 10,000$  feet;  
 $\dot{x}_1 = -500$  feet per second.

Figure 9.- The effect of gravitation on the ferry's motion relative to satellite in a circular orbit at an altitude of 500 statute miles for  $\zeta = 0.7$  and  $\omega_0 = 0.04$  per second.



(e)  $x_1 = 100,000$  feet;  
 $\dot{x}_1 = -500$  feet per second.

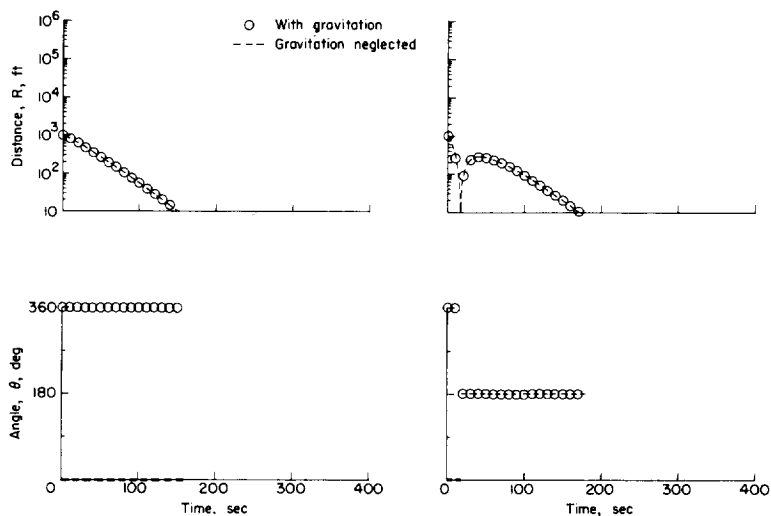
(f)  $x_1 = 100,000$  feet;  
 $\dot{x}_1 = -1,000$  feet per second.



(g)  $x_1 = 1,000,000$  feet;  
 $\dot{x}_1 = -1,000$  feet per second.

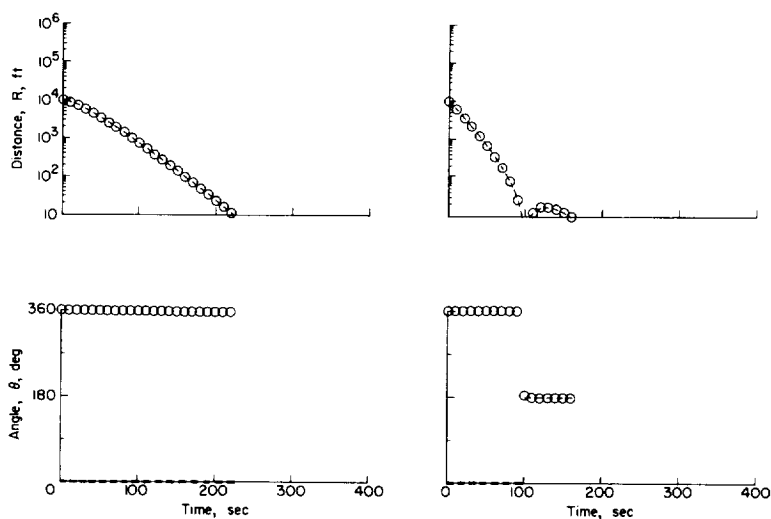
(h)  $x_1 = 1,000,000$  feet;  
 $\dot{x}_1 = -2,000$  feet per second.

Figure 10.- Concluded.



(a)  $x_1 = 1,000$  feet;  
 $\dot{x}_1 = -20$  feet per second.

(b)  $x_1 = 1,000$  feet;  
 $\dot{x}_1 = -100$  feet per second.



(c)  $x_1 = 10,000$  feet;  
 $\dot{x}_1 = -100$  feet per second.

(d)  $x_1 = 10,000$  feet;  
 $\dot{x}_1 = -500$  feet per second.

Figure 11.- The effect of gravitation on the ferry's motion relative to a satellite in a circular orbit at an altitude of 500 statute miles for  $\zeta = 1.0$  and  $\omega_0 = 0.04$  per second.

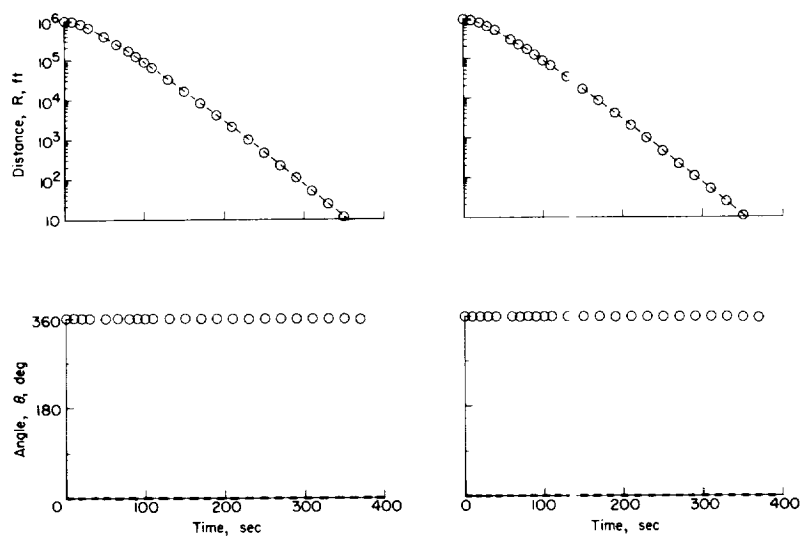
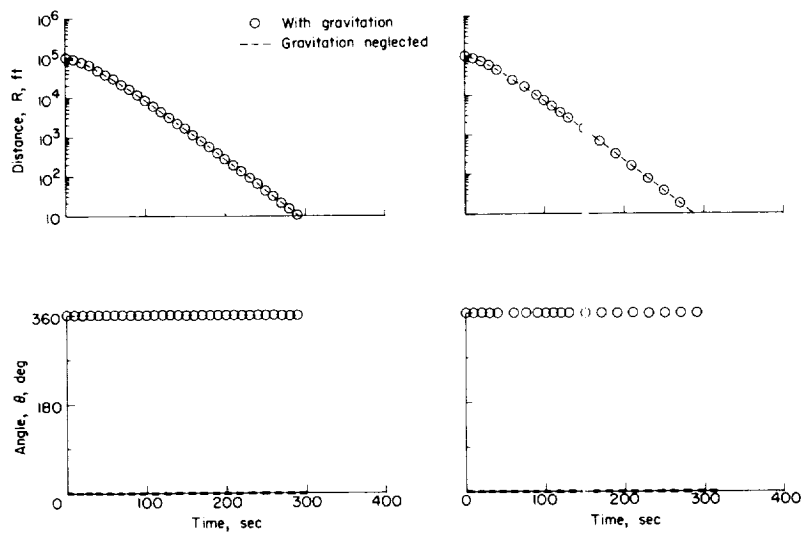
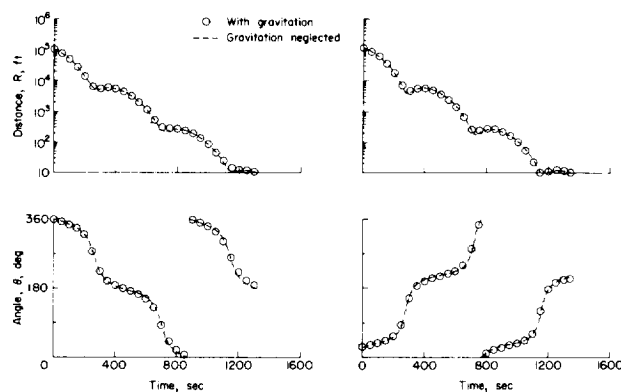


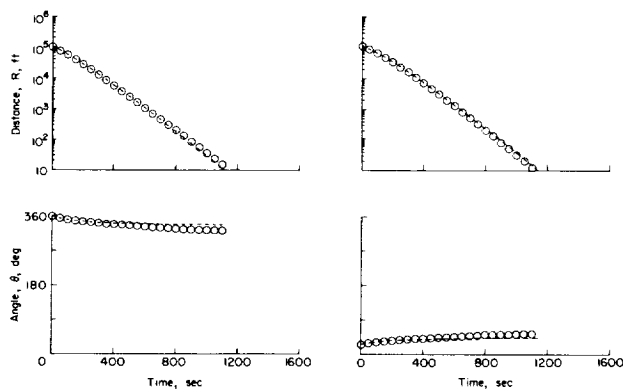
Figure 11.- Conclude 1.





(a)  $x_1 = 100,000$  feet;  
 $\dot{x}_1 = -500$  feet per second;  
 $x_2 = 0$ ;  
 $\dot{x}_2 = -250$  feet per second;  
 $\zeta = 0.7$ .

(b)  $x_1 = 100,000$  feet;  
 $\dot{x}_1 = -500$  feet per second;  
 $x_2 = 50,000$  feet;  $\dot{x}_2 = 0$ ;  
 $\zeta = 0.7$ .



(c)  $x_1 = 100,000$  feet;  
 $\dot{x}_1 = -500$  feet per second;  
 $x_2 = 0$ ;  
 $\dot{x}_2 = -250$  feet per second;  
 $\zeta = 1.0$ .

(d)  $x_1 = 100,000$  feet;  
 $\dot{x}_1 = -500$  feet per second;  
 $x_2 = 50,000$  feet;  
 $\dot{x}_2 = 0$ ;  $\zeta = 1.0$ .

Figure 12.- The effect of gravitation on the ferry's motion relative to a satellite in a circular orbit at an altitude of 500 statute miles for  $\omega_0 = 0.01$  per second.

